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BEYOND H_0 AND Q_0: COSMOLOGY IS NO LONGER JUST TWO NUMBERS

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ABSTRACT

For decades, H_0 and q_0 were the quest of cosmology, as they promised to characterize our "world model" in a model-independent way. Using simulated data, we show that q_0 *cannot* be both accurately and precisely determined using distance indicators. While H_0 *can* be both accurately and precisely determined, to avoid a small bias in its direct measurements (of order −5%) we demonstrate that H_0/Ω_M (assuming flatness and $w = -1$) is a better choice of two parameters, even if our world model is not precisely ΛCDM. We illustrate with the analysis of the Constitution set of supernovae and indirectly infer $q_0 = -0.57 \pm -0.04$. Finally, we show that it may be possible to directly determine q_0 using the time dependence of redshifts, a method far less susceptible to the biases that plague measurements using distance indicators.

Subject headings: cosmological parameters — methods: numerical — supernovae: general

1. INTRODUCTION

Allan Sandage once characterized cosmology as the search for two numbers (Sandage 1970): H_0, the present expansion rate, and q_0, the present deceleration parameter. Few would argue with the statement that cosmology is a much grander enterprise today and that our "world model" is better described by a larger set of physically motivated parameters (including the energy densities of radiation, dark matter, dark energy, and the equation-of-state of dark energy). In this paper we show that in fact, the deceleration parameter cannot be both accurately and precisely measured. While the expansion rate can be, avoiding a biased measurement requires a better choice of parameters than q_0 and H_0, e.g., Ω_M and H_0.

Sandage introduced H_0 and q_0 to provide a model-independent, kinematic description of the expansion of the Universe; this description begins with a Taylor series for the cosmic scale factor $R(t)$,

$$R(t)/R_0 = 1 + H_0(t - t_0) + \frac{1}{2}q_0 H_0^2(t - t_0)^2 + \cdots \quad (1)$$

Where $H_0 \equiv \dot{R}_0/R_0$ and $q_0 \equiv -(\ddot{R}_0/R_0)/H_0^2$. Using the definition of redshift, $1 + z \equiv R_0/R$, and luminosity distance, $d_L \equiv (1 + z)r(z)R_0$, and the above expansion, the observable luminosity distance can be expressed in a Taylor series in redshift

$$H_0 d_L = z + \frac{1}{2}(1 - q_0)z^2 + O(z^3) \quad (2)$$

Note too, that no assumption about the validity of general relativity has been made; only that spacetime is isotropic and homogeneous and described by a metric theory. We note that the next order term–the jerk parameter j_0–may be added (Visser 2004; Chiba & Nakamura 1998):

$$R(t)/R_0 = 1+H_0(t-t_0)-\frac{1}{2}q_0H_0^2(t-t_0)^2+\frac{1}{6}j_0H_0^3(t-t_0)^3+\cdots$$

$$H_0 d_L = z+\frac{1}{2}(1-q_0)z^2-\frac{1}{6}(1-q_0-3q_0^2+j_0+R_{\rm curv}^{-2}H_0^{-2})z^3+\cdots$$

where $j_0 \equiv (\dddot{R}_0/R_0)/H_0^3$ and $R_{\rm curv}$ is the 3-curvature (radius) of the Universe.

The power of the H_0/q_0 (or quadratic) expansion is that, in principle, measurements of $d_L(z)$ can be used to determine the present expansion rate–arguably the most important number in all of cosmology–and the deceleration parameter. Moreover, in the simple matter-only cosmology of the time, General Relativity, through the Friedmann equations, relates q_0 to the physical parameter, $\Omega_0 \equiv \rho_M/\rho_{\rm crit}$ (ρ_M is the matter density and $\rho_{\rm crit} \equiv 3H^2/8\pi G$ is the critical density):

$$q_0 = \Omega_0/2$$

Further, in this model, q_0 is related to the spatial curvature and destiny of the Universe: if q_0 is greater than $1/2$, Ω_0 is greater than unity and the Universe is positively curved and will ultimately re-collapse; conversely, if q_0 is less than $1/2$, Ω_0 is less than 1 and the Universe is negatively curved and will expand forever. The case of $q_0 = 1/2$ is the uncurved (flat) Universe that expands forever at an ever slowing rate.

This is all well and good; the question we address here is whether or not these two (or three), model-independent parameters can in fact be determined with accuracy and precision. The answer is simple: only H_0 can be measured with precision and accuracy. The explanation is simple as well: at low redshifts, say $z \lesssim 0.2$,

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where the Taylor expansion is most accurate, poor leverage on q_0 (and j_0) and peculiar velocities severely limit the precision; at higher redshifts, where the effect of peculiar velocities is negligible and the leverage is greater, the quadratic expansion does not accurately approximate $d_L(z)$ (see Fig. 1) and a bias is introduced in the measurement. For q_0, you can have accuracy (restrict measurements to low redshift) or precision (use high redshift measurements), but not both! (We also show that including the cubic term in the model-independent expansion–jerk j_0–does change this conclusion.)

Moreover, while one might be tempted to simply abandon q_0 and j_0 given these complications, in fact they have found use recently to avoid a bias of order -5% that otherwise inflicts direct measurements of H_0 (due to the less accurate linear expansion, $H_0 d_L = z$) (Riess et al. 2009, 2011). To better avoid such biases, we explore other two- or three-parameter descriptions of d_L. In particular, the two parameters H_0 and Ω_M (where flatness and $w = -1$ are assumed), eliminates this small bias in measuring H_0 and are better motivated. We use the Constitution compilation of supernovae (Hicken et al. 2009b) to illustrate our findings. Finally, we speculate about measuring q_0 by using the very small time variation of redshifts (cm/sec/yr).

Our paper is organized as follows. In Section 2 we detail our distance indicator simulations and mock supernova data. We present the results of these simulations in Section 3, including the use of the Constitution supernova dataset to provide concrete illustrations of our findings. Section 4 provides some discussion and conclusions.

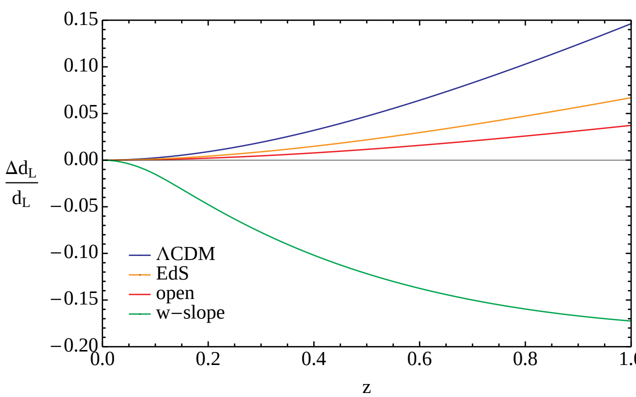

Figure 1. Fractional error $\Delta d_L/d_L$ of the quadratic expansion for d_L in various cosmologies. Note that $\Delta\mu \approx 2.17\Delta d_L/d_L$, where μ is the distance modulus and Δd_L and $\Delta\mu$ are the absolute errors of the quadratic expansions for d_L and μ, respectively.

2. SIMULATIONS

For a diverse set of cosmological models (Table 1) we generate mock distance modulus and redshift (μ, z) data over various redshift ranges, then study the degree to which q_0 and j_0 are constrained when these data are analyzed with the quadratic (H_0/q_0) and cubic $(H_0/q_0/j_0)$ expansions. We then compare the degree to which H_0 is constrained using these expansions to the constraints obtained when the analysis assumes ΛCDM. In the latter case, H_0 and Ω_M are the free parameters. We also investigate the effect of peculiar velocities and intrinsic luminosity scatter.

Table 1
Cosmological models used in our simulations.

Model	Ω_M	Ω_{DE}	$w_{DE}(z)$	q_0
ΛCDM	0.28[a]	0.72[a]	-1	-0.58
de Sitter	0	1	-1	-1
open	0.28	0	–	0.14
Einstein-de Sitter	1	0	–	0.5
w-slope	0.28	0.72	$\begin{cases} -10z & \text{for } z < 0.1 \\ -1 & \text{for } z > 0.1 \end{cases}$	0.5

[a]Komatsu et al. (2011).

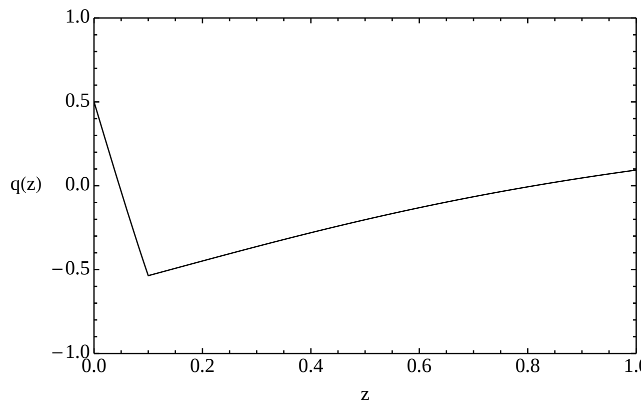

Figure 2. Deceleration parameter, $q(z)$, in the w-slope model (see Table 1).

2.1. *Mock Distance Indicators*

We model the distance indicators as a population of imperfect standard candles with an intrinsic Gaussian absolute magnitude scatter $\sigma_{\rm int}$. (While we are motivated by and will eventually use type Ia SNe as the distance indicator, our results are more general.) We explore two possible values of $\sigma_{\rm int}$: 0.15 mag, today's state of the art for SNe Ia after light curve fitting (e.g. Kowalski et al. 2008; Kessler et al. 2009; Hicken et al. 2009a; Rapetti et al. 2007); and 0.02 mag, a low level at which peculiar velocity scatter dominates the μ uncertainties at $z \lesssim 0.2$. Note the intrinsic distance modulus scatter is simply related to the luminosity scatter as $\sigma_{\rm int} = (d\mu/dL)\sigma_L = 1.08(\sigma_L/L)$.

Each 'measured' μ is chosen from a Gaussian distribution of width $\sigma_{\rm int}$ centered on the true value of $\mu(z)$, which is given by standard equations (Weinberg 2008). We then apply a modest peculiar velocity scatter to the redshift data, giving the 'measured' redshift: $(z+1)(z_{\rm pec}+1)-1$, where each $z_{\rm pec}$ is chosen from a Gaussian distribution centered at zero with $\sigma_{\rm z,pec} = 0.001$ (300 km/s) (e.g. Riess et al. 2004, 2007; Rapetti et al. 2007; Hicken et al. 2009b; Lampeitl et al. 2010). [We note that there is no a priori reason why peculiar velocities should be described by an independent redshift scatter applied to each standard candle, and in general, one would expect correlations due to bulk flows which would prevent statistical uncertainties in the the fit parameters from decreasing as fast as $1/\sqrt{N}$, where N

is the number of data points. However, previous surveys (e.g. Lampeitl et al. 2010) and N-body simulations (Vanderveld 2008) have suggested that such correlation is not significant enough to pose a large issue.]

In parameter estimations, this peculiar velocity scatter can be incorporated as an additional uncertainty in μ added in quadrature to the intrinsic scatter: $(d\mu/dz)\sigma_{z,pec} = 2.17\sigma_{z,pec}/z$, approximating $\mu(z)$ with the linear expansion for d_L. While others have assumed the empty universe model here (e.g. Kessler et al. 2009; Lampeitl et al. 2010), essentially taking the quadratic expansion with $q_0 = 0$, we found the difference on our parameter estimates to be negligible. Figure 3 shows the overall μ scatter $\sigma_\mu(z)$,

$$\sigma_\mu(z) = \sqrt{\sigma_{\rm int}^2 + (2.17\sigma_{z,pec}/z)^2} \tag{3}$$

as well as the μ uncertainties in the Constitution data set (Hicken et al. 2009b) with the intrinsic scatter $\sigma_{\rm int} = 0.15$ mag added in quadrature.

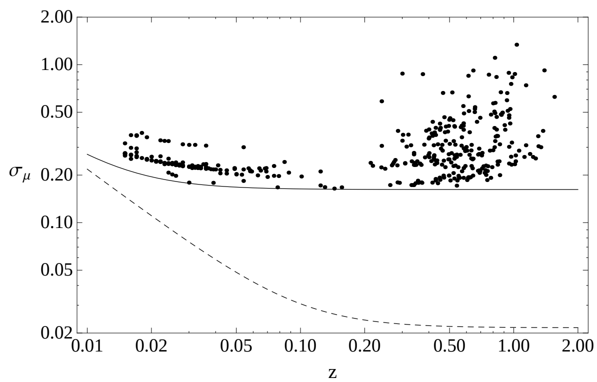

Figure 3. Overall distance modulus uncertainty $\sigma_\mu(z)$ (Eq. 3) used in our simulations with $\sigma_{\rm int} = 0.15$ (solid) and $\sigma_{\rm int} = 0.02$ (dashed) (including intrinsic scatter and peculiar velocity scatter), as well as the uncertainties in the Constitution data with $\sigma_{\rm int} = 0.15$ added in quadrature.

2.2. *Mock Distance Indicator Surveys*

For a given cosmological model, redshift range, and intrinsic standard candle scatter, we simulated 10,000 surveys, each with 500 mock objects (see Section 2.1) uniformly distributed over redshift.[1] We use a minimum redshift of 0.015[2] and explore several values of the maximum redshift chosen to best illustrate the compromise between accuracy at low redshift, and precision at high redshift. In our quadratic expansion studies, we use $z_{\rm max} = 0.1, 0.3, 0.5$. When we explore the cubic expansion, we use $z_{\rm max} = 0.7, 0.9, 1.5$, as this expansion remains accurate until these higher redshifts. Redshifts smaller that 0.015 are unhelpful for cosmology as peculiar velocity uncertainties are too crippling for such nearby objects (see Fig. 3). To eliminate simulation biases due to mock redshifts scattering *out* of the redshift range $(0.015, z_{\rm max})$, but not scattering *into* it, we actually generate more than 500 data points over a slightly larger redshift range, then kept only 500 data points in the desired range.

We then explore biases in precision measurements of H_0 due to the inaccuracy of the d_L expansion. To avoid any systematics present in high-z supernova data (such as evolution or dust effects), typically only nearby supernovae are used to measure H_0. For these simulations, we take $z_{\rm max} = 0.1$, which can be identified as the cutoff redshift of the low-z segment of the Constitution set (Fig. 3).

2.3. *Parameter Estimation*

For each mock survey, we estimate model parameters by minimizing a χ^2 statistic. For instance, to estimate H_0 and q_0 using the quadratic expansion, we minimize:

$$\chi^2(H_0, q_0) = \sum_{i=1}^{500} \left(\frac{\mu_i - \mu_{\rm exp}(H_0, q_0; z_i)}{\sigma_\mu(z_i)} \right)^2 \tag{4}$$

Then for each cosmological model, redshift range, and intrinsic μ scatter, we plot 2D contours in H_0/q_0 parameter space with 68% and 95% probability content generated from the 10,000 parameter estimations. We treat the 10,000 (H_0,q_0) pairs as samples from a 2D Gaussian, compute the covariance matrix, then plot contours of that Gaussian. We have verified that the error in assuming the distribution is Gaussian is not significant for our purposes. Each contour plot shows the probability distribution of (H_0,q_0) pairs in parameter space given the intrinsic μ scatter and peculiar velocity scatter the data are drawn from, showing any bias in and correlation between the parameter estimates. The extents of the contours may be taken as error ellipses for a single measurement.

To estimate q_0 and j_0 using the cubic expansion, we minimize $\chi^2(H_0, q_0, j_0)$ function, then marginalize over H_0 and proceed as discussed.

When analyzing the Constitution data set, we pick several illustrative maximum redshifts, though unlike with our mock data sets, the number of objects is not the same for each redshift range. The numbers of SNe up to $z_{\rm max} = \{0.1, 0.3, 0.5, 1.5\}$ are {141, 164, 248, 397}. Following standard practice (e.g. Lampeitl et al. 2010; Kessler et al. 2009; Rapetti et al. 2007), we add the intrinsic standard candle scatter (0.15 mag) in quadrature to the published μ uncertainties.

3. RESULTS

Figure 4 shows H_0/q_0 contour plots from analysis of 10,000 mock ΛCDM simulations with the quadratic expansion, as well as contours from analysis of Constitution data. The trend with $z_{\rm max}$ (and our basic result) is clearly illustrated: increasing $z_{\rm max}$ increases the precision and decreases the accuracy. Further, the precision at small $z_{\rm max}$ is fundamentally limited by the scatter due to peculiar velocities. This basic trend is seen for all the cosmological models (except deSitter, see below).

Shrinking the intrinsic μ scatter to 0.02 mag (the regime where the parameter uncertainties in the $z_{\rm max}$ =

[1] In theory, there is a great deal of astrophysics and cosmology at play in determining the redshift dependence of number density of a class of distance indicators. In practice, observational challenges (e.g. Li, Filippenko, & Riess 2001) often bury even those effects. For simplicity and to illustrate our basic results, we use a uniform distribution over redshift with different maximum redshifts.

[2] This is the smallest redshift present in the Constitution set.

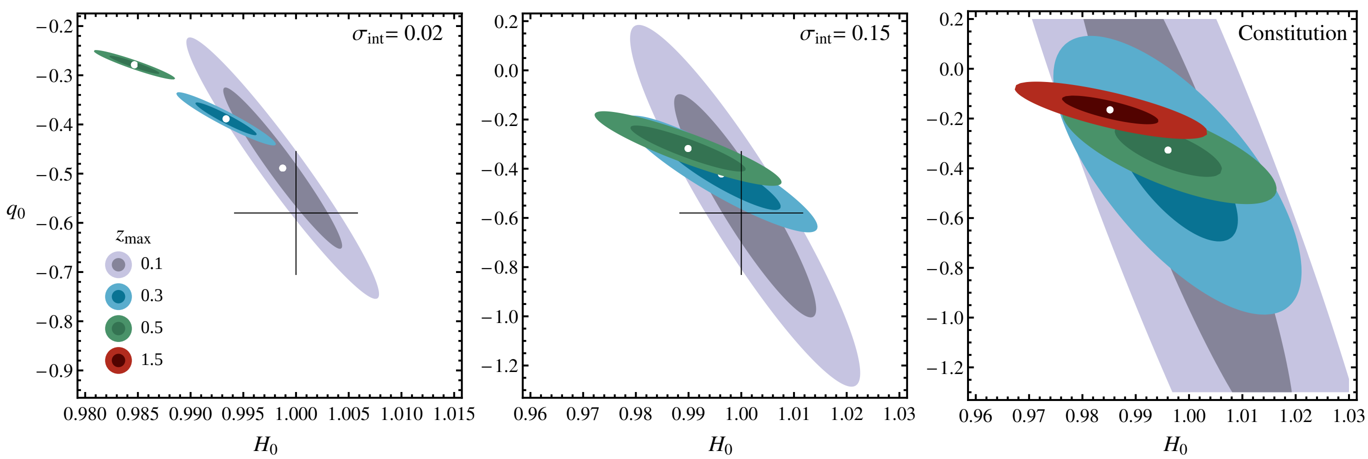

Figure 4. q_0/H_0 contours from ΛCDM simulations analyzed with quadratic expansion, and contours from analysis of Constitution data. The crosshairs indicate the true $(H_0, q_0) = (1, -0.58)$ in ΛCDM, where H_0 is in units of its true value. The compromise between accuracy at low redshift, and precision at high redshift is evident both in the simulations and in the Constitution data. Constitution H_0 values are in units of 65 km s^{-1} Mpc^{-1}, as used in (Hicken et al. 2009b). Note the expanded axes of the left-most plot.

0.1 simulations are dominated by peculiar velocity scatter), reduces the q_0 uncertainties by a factor of $\sim$4, though the biases worsen slightly due to the altered distribution of uncertainties as a function of z (high z data now weighted much more relative to low z data, see Fig. 3). We note here that while our simulations with $\sigma_{\rm int} = 0.15$ mag are intended to match state of the art SNe Ia surveys, the Constitution data set contains many objects with much larger uncertainties and the set is distributed non-uniformly over redshift (Fig. 3). Thus while the Constitution H_0/q_0 contours exhibit the same trends as those in the $\sigma_{\rm int} = 0.15$ simulation plot, the Constitution uncertainties are larger.

Similar contour plots for other cosmological models (Table 1) appear in Fig. 5. Again we observe the expected trade-off between accuracy at low z, and precision at high z. The open and EdS models are somewhat better approximated by their quadratic expansions than is ΛCDM (Fig. 1), and thus, they admit slightly more accurate q_0 estimates. Estimates of q_0 in the w-slope model are especially poor owing to its rapid evolution over redshift (Fig. 2). The de Sitter model is worth mentioning because it provides a check on the reliability of the Monte Carlo simulations. In that model, the quadratic expansion is exact, a fact reflected in our parameter estimates by the absence of any biases.

Figure 6 shows the results of including the cubic term when analyzing data from the ΛCDM model in the form of j_0/q_0 contour plots after marginalizing over H_0. The overall trend is the same as before: increasing $z_{\rm max}$ increases precision and decreases accuracy. Relative to the quadratic expansion, the accuracy is improved and we thus use $z_{\rm max} = 0.7, 0.9, 1.5$, and note that over the 3 redshift ranges previously considered ($z_{\rm max} = 0.1, 0, 3, 0.5$), the uncertainties in the q_0 estimates increase by a factor of $\sim$5. Reducing $\sigma_{\rm int}$ to 0.02 mag helps somewhat, but as before worsens the biases slightly.

Using the entire Constitution set ($z_{\rm max} = 1.5$), we estimate $q_0 = -0.64 \pm 0.14$, which deviates *negatively* from the true ΛCDM value of -0.58, while the simulation result (with same $z_{\rm max}$ and $\sigma_{\rm int} = 0.15$) of -0.45 ± 0.09 deviates *positively*. To explain this, we first note again that the Constitution data are distributed non-uniformly over redshift. In fact, 80% of the data are below $z = 0.8$, and nearly all the $z > 0.2$ data are not limited by the 0.15 mag intrinsic scatter (Fig. 3). Given these facts and only a single data realization (as opposed to the 10,000 realizations the simulation plots average over), one expects fluctuations in the $z_{\rm max} = 1.5$ (red) Constitution contour position roughly within the $z_{\rm max} = 0.7$ (green) contour. To check that this is the case, we performed 10,000 ΛCDM simulations using the Constitution redshifts and μ uncertainties. The resulting contour is plotted in black has $q_0 = -0.52 \pm 0.14$, and matches better the q_0/j_0 biases observed in the simulations. It appears slightly larger than the $z_{\rm max} = 1.5$ (red) Constitution contour because the actual distribution is slightly non-Gaussian.

Next, we explore the small biases that creep into precision H_0 measurements due to the inaccuracy of the d_L expansion. We simulate distance indicator surveys with $z_{\rm max} = 0.1$ (as discussed in Sec. 2.2) in ΛCDM and in two models close to ΛCDM but still consistent with current data (ΛCDM may not be the exact world model): a model with $w = -0.8$ that is otherwise identical to ΛCDM; and a slightly closed model with $w = -1$, $\Omega_M = 0.30$, and $\Omega_{DE} = 0.75$ ($\Omega_K = -0.05$). We then estimate H_0 by fitting to:

1. H_0/Ω_M parameterization of ΛCDM (assuming flatness and $w = -1$)

2. Quadratic expansion, marginalizing over q_0

3. Quadratic expansion, fixing $q_0 = -0.58$

4. Cubic expansion, marginalizing over q_0 and j_0

5. Cubic expansion, fixing $q_0 = -0.58$ and $j_0 = 1$

The results are shown in Fig. 7.

We note that using the linear expansion for d_L yields a bias in H_0 (not shown) of -4% for $\sigma_{\rm int} = 0.15$, and -5% for $\sigma_{\rm int} = 0.02$, in good agreement with with the -3% bias identified by Riess et al. (2009) using real data. Using the quadratic or cubic expansion and marginalizing

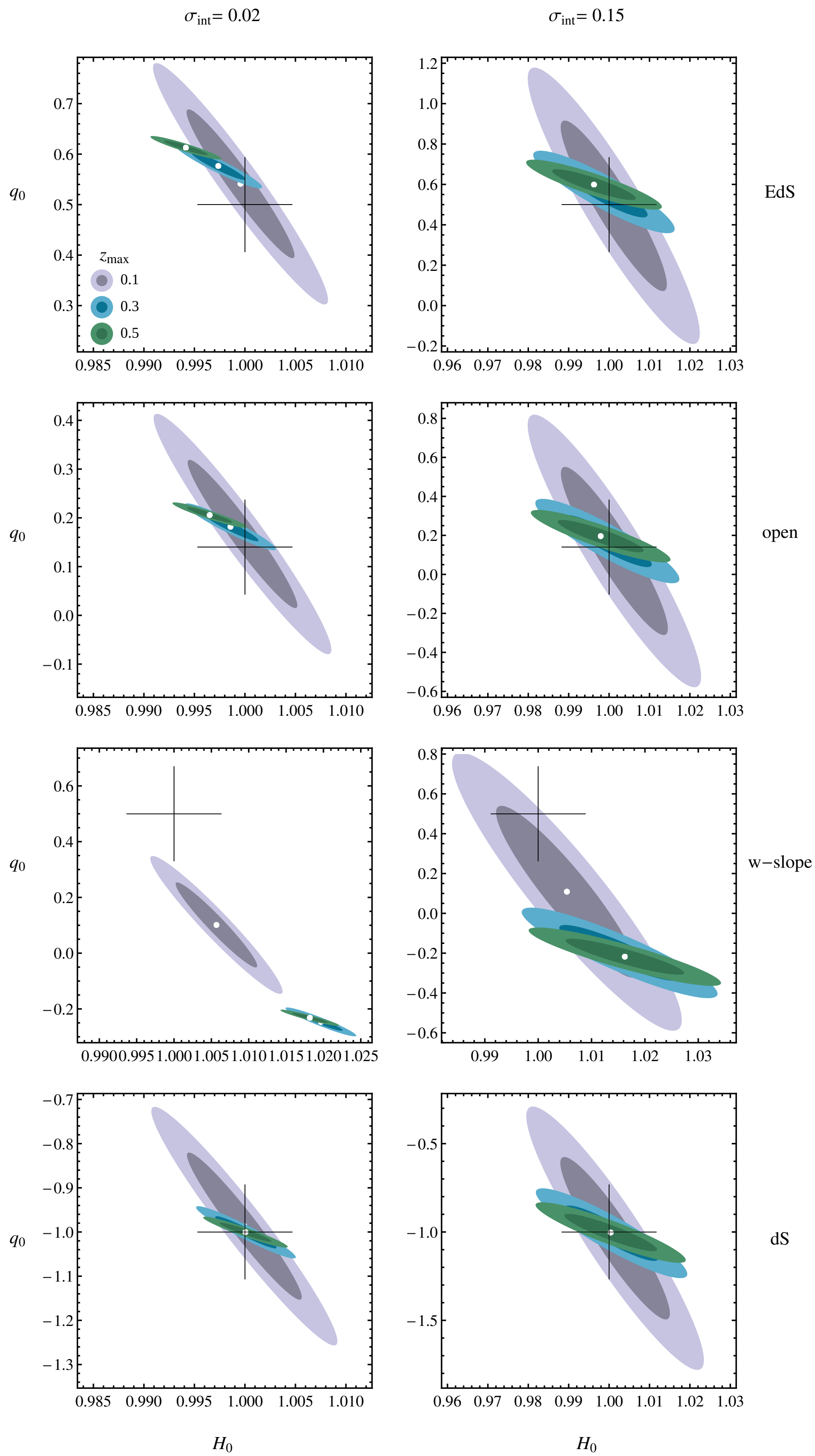

Figure 5. q_0/H_0 contours from simulations in various cosmological models (Table 1) analyzed with quadratic expansion. The crosshairs indicate the true (H_0,q_0) in each model, where H_0 is in units of its true value. The compromise between accuracy at low redshift, and precision at high redshift is visible in all models except the de Sitter model, in which the quadratic expansion is exact. Estimates of q_0 in the w-slope model are especially poor owing to its rapid evolution over redshift (Fig. 2). Note the expanded axes in the left column.

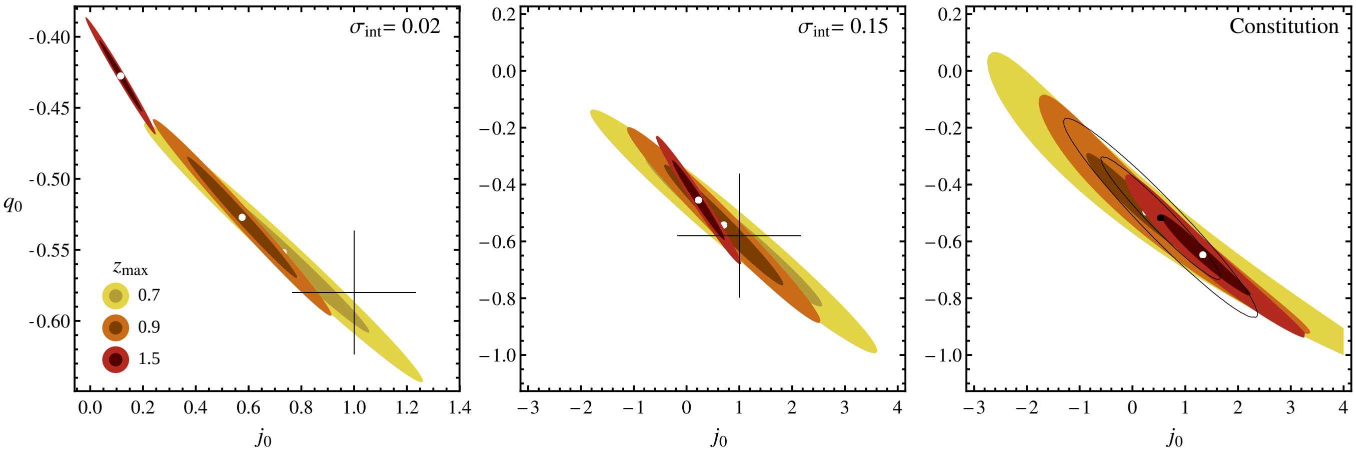

Figure 6. q_0/j_0 contours from ΛCDM simulations analyzed with cubic expansion (after marginalizing over H_0), as well as contours from analysis of Constitution data. The crosshairs indicate the true $(j_0, q_0) = (1, -0.58)$ in ΛCDM. As the accuracy of the parameter estimates is improved (relative to estimates using quadratic expansion), we show the results of simulations over larger redshift ranges where the compromise between precision and accuracy is clearer. Non-uniformly spaced redshift data and larger high-z μ uncertainties explain the discrepancy between the Constitution ($z_{\rm max} = 1.5$) q_0 estimate and the $\sigma_{\rm int} = 0.15$ simulation result. See text for discussion. The black contours in the Constitution plot are the result of performing 10,000 simulations with Constitution redshifts and μ uncertainties assuming ΛCDM. Note the expanded axes of the left-most plot.

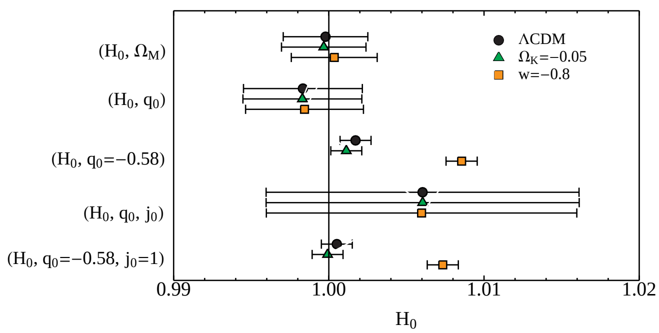

Figure 7. H_0 estimates from ΛCDM (and models close to ΛCDM) obtained by fitting $\sigma_{\rm int} = 0.02$ mock data to 5 different models using $z_{\rm max} = 0.1$. With $\sigma_{\rm int} = 0.15$, H_0 uncertainties are a factor of 2–3 larger and biases are comparable.

over the other parameters reduces these biases to order 0.5%, and assuming values for those parameters reduces them further (except for the $w = -0.8$ model).

Even if our world model is not ΛCDM, the H_0/Ω_M parameterization is the most robust, with biases of order 0.01%. With $\sigma_{\rm int} = 0.02$, peculiar velocities dominate over μ uncertainties at these redshifts, and provide the limiting uncertainty in H_0. They are also the source of the lingering $\sim$ 0.01% biases, as the peculiar velocity uncertainties are estimated using *measured* z, not actual z, so objects scattering to lower (higher) z are under- (over-) weighted in the likelihood analysis.

As a final check on the reliability of our simulations, Fig. 8 shows the analysis of ΛCDM simulations and of Constitution data with the H_0/Ω_M parameterization. Our simulations show no biases and from this Constitution data we determine $q_0 = 3\Omega_M/2 - 1 = -0.57 \pm 0.04$, with better precision *and* accuracy than estimates using the quadratic or cubic expansions.

4. DISCUSSION

The quest to measure H_0 and q_0 and determine our world model drove cosmology for almost three decades. The Hubble constant has now been directly measured to around five percent precision (Riess et al. 2011), inferred from CMB anisotropy and other measurements to almost 1% (Komatsu et al. 2011), and there are aspirations to improve that to less than a percent (Riess et al. 2011; Suyu et al. 2012). H_0 remains arguably the most important single number in cosmology, as it sets the age and size of the Universe and underpins many other measurements.

On the other hand, not only was the minus sign in the definition of q_0 unfounded, but also q_0 cannot actually be measured! The reasons are simple: at low redshifts, where the d_L expansion is accurate, the combination of peculiar velocities and the small change in d_L for different values of q_0 (lack of leverage) makes q_0 impossible to measure with any precision. At higher redshift, where there leverage and precision is possible, the inaccuracy of the H_0/q_0 expansion leads to bias in determining q_0. There is no 'sweet spot' at intermediate redshift that allows both accuracy and precision regardless of the world model.

Moreover, the mere usage of q_0 as the second parameter for precision measurements of H_0 leads to a non-negligible bias. As we and others have shown (Riess et al. 2009), fitting to a linear Hubble law introduces a systematic bias of order -5%. One solution, used by Riess et al. (2009, 2011), is to use the $H_0/q_0/j_0$ expansion and by fiat impose the 'correct' values of q_0 and j_0 obtained with $z \sim 1$ data. However, despite the fact that H_0 estimates using this method are *no longer independent of high-z supernovae*, this method is not robust to small changes in w, acquiring a $\sim$ 1% bias if our world model actually has $w = -0.8$ (Fig. 7), within the error bars of recent measurements (Hicken et al. 2009b; Komatsu et al. 2011). Further, neglecting the uncertainties on q_0 and j_0 priors *underestimates* the real uncertainty in H_0, providing a false sense of accuracy. In our simulations, going from *fixing* those values to *freeing* them increases the H_0 uncertainty by $\sim$ 1%, an amount that will be important in future precision attempts to further constrain H_0 using this method.

Instead and what we think is better, one can use the

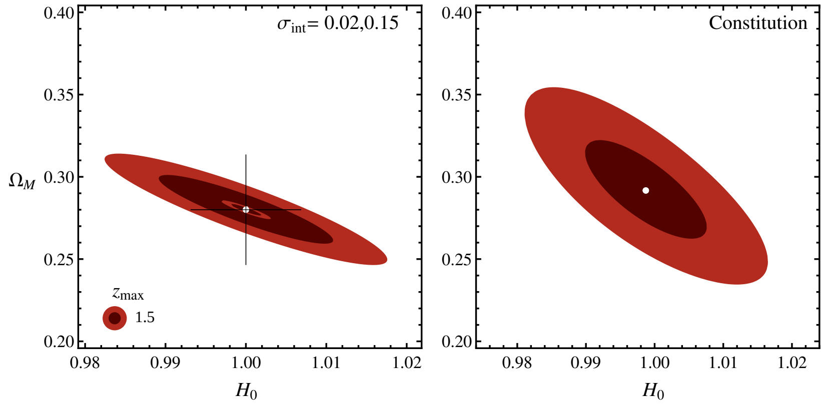

Figure 8. H_0/Ω_M contours from ΛCDM simulations analyzed with the standard H_0/Ω_M parametrization, plotted with H_0/Ω_M contours from same analysis on Constitution data. The crosshairs indicate the true simulation values $(H_0, \Omega_M) = (1, 0.28)$. Simulation H_0 values in units of true H_0, Constitution H_0 values in units of 65 km s^{-1} Mpc^{-1}, as used in (Hicken et al. 2009b). Note the absence of any large bias in either parameter.

more physical two-parameter description of our cosmological model, H_0/Ω_M. If our world model is indeed ΛCDM, then these are the only two parameters needed to describe $d_L(z)$. In this case, there is essentially no bias in determining the Hubble constant and only peculiar velocities limit the precision (Fig. 7). The same figure shows that even if ΛCDM does not exactly describe our world model, the biases are still very small.

Additional parameters can be added (and may be needed) to completely characterize d_L: for example, w or w_0 and w_a, and Ω_k, but given the closeness of our world model to ΛCDM, H_0/Ω_M is a set of two parameters that affords both accuracy and precision. Further, we note that within ΛCDM additional parameters are needed to characterize CMB anisotropy and other dynamical aspects of the Universe. The standard 6-parameter set is $\Omega_B h^2$ (baryon density), $\Omega_M h^2$ (total matter density), Ω_{DE}, n_S (power-law index of density perturbations), τ (optical depth to last scattering), and Δ^2 (overall amplitude of the spectrum of inhomogeneity); (e.g. Komatsu et al. 2011), and even more parameters can be added. Cosmology today is much richer than the two numbers that Sandage used to characterize our world model and the goal of cosmology.

All this being said, there is a certain elegance and utility to the two parameters H_0 and q_0, even if one of them is difficult to directly measure. For example, both can be defined independently of general relativity and its Friedmann equations [all that is required is the assumption of isotropy and homogeneity and a metric theory (Turner & Riess 2002; Shapiro & Turner 2006)]. Together, they characterize the most basic features of the expansion – rate of expansion and whether the expansion is slowing down or speeding up. Within the parameterizations mentioned above, q_0 can be inferred; for example, with the two parameters Ω_M and H_0:

$$q_0 = \frac{3}{2}\Omega_M - 1 = -0.57 \pm 0.04$$

where the numerical value has been determined from the Constitution data set.

Finally, we mention a possibility for directly measuring q_0 with small bias: the time dependence of cosmological redshifts. The redshift of an object within the Hubble flow varies with time due to the expansion of the Universe (Sandage 1962; Loeb 1998) (for an overview see Liske et al. 2008, and refs. therein):

$$\frac{dz}{dt} = (1+z)H_0 - H(z) \longrightarrow -zq_0H_0 + O(z^2)$$

where the H_0/q_0 expansion has been used to obtain the final expression. This method too is subject to bias if z is not small. However, the potential observable, dz/dt, depends upon the product of H_0 and q_0 (meaning H_0 must be determined separately) and is linear in redshift (not quadratic). Of course, this is a very challenging measurement–the effect is on the order of a few cm sec^{-1} decade^{-1}–and a reach goal for the next generation of extremely large optical telescopes; nonetheless we explore it further for directly measuring q_0.

The estimate for q_0 from this method, assuming H_0 has been determined by other means, is given by

$$q_0^{\rm est} = -\frac{1}{H_0 z}\frac{dz}{dt} + O(z) \qquad (5)$$

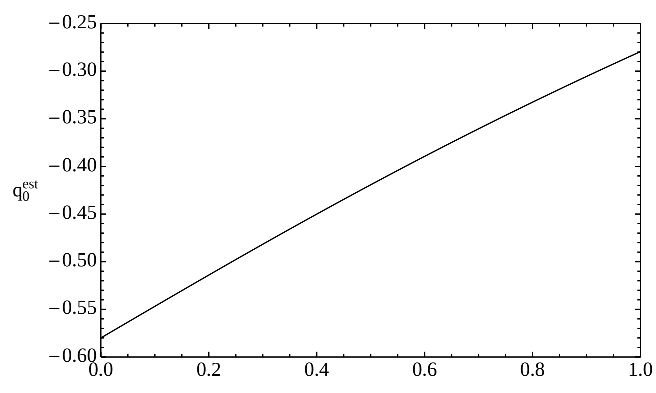

Figure 9. Estimate of q_0 (Eq. 5) obtained from dz/dt measurement.

Fig. 9 shows $q_0^{\rm est}$ as a function of the redshift z of the object whose time variation is being measured, assuming that the correct cosmology is ΛCDM and that H_0 is known well. For z less than 0.1, the bias is less than

5% ($\Delta q_0 \sim +0.03$), and the precision is limited *only* by that of the dz/dt measurement. (Such accuracy is possible when using luminosity distances, but only at the expense of large imprecision, $\Delta q_0 \sim 1$, cf., Fig. 4.)

Cosmology has changed dramatically since Sandage characterized it as the quest for two numbers. It has become a precision science characterized by a larger set of more physically motivated numbers. While, q_0 is not actually measurable using luminosity distance and a hindrance to accurately measuring H_0, it is nonetheless of interesting parameter in cosmology today–and still not directly measured.

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